Lecture 32

NC and AC: Subclasses of $P_{/poly}$

Depth of a Circuit

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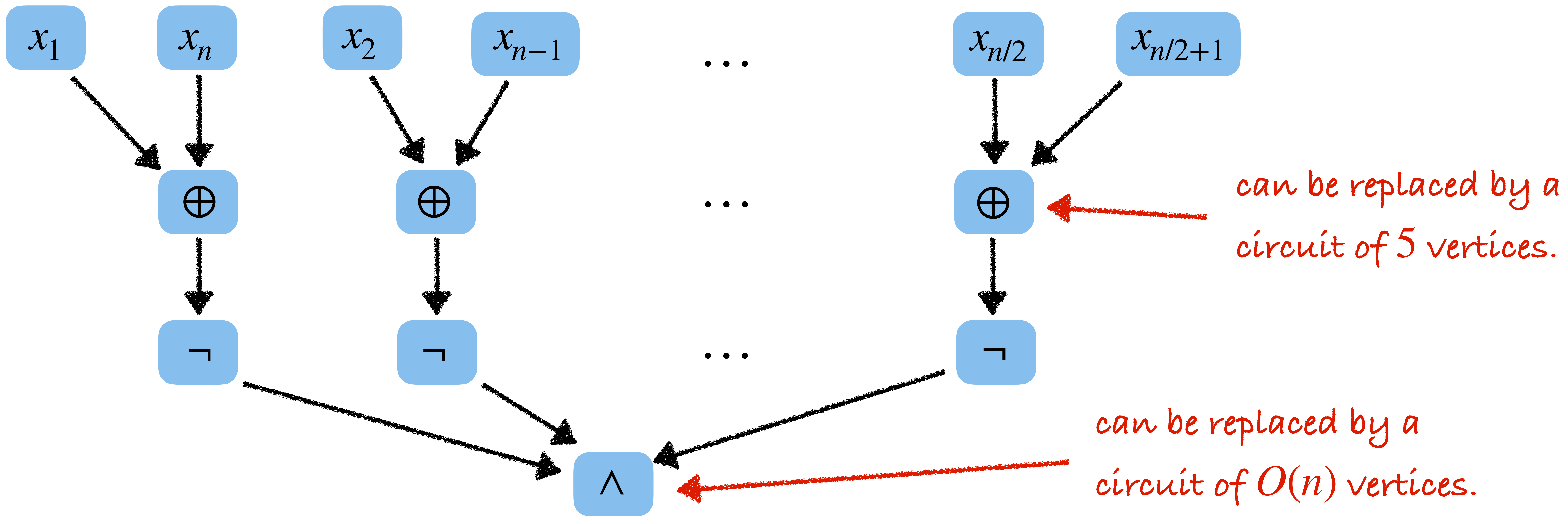

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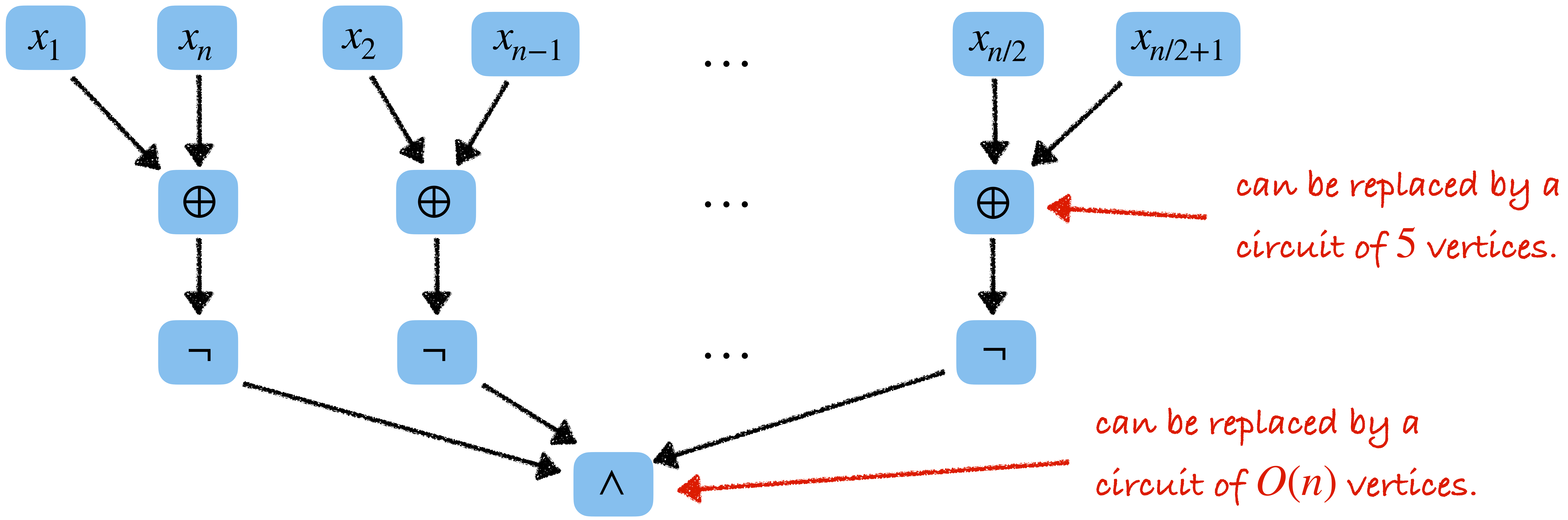

Example: Depth of *PALIN*'s circuit was $O(\log n)$.

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Definition: **uniform-NC** and **uniform-AC** require circuits to be logspace uniform.

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- Showing $\mathbf{NP} \not\subseteq \mathbf{NC}$ might be easier than $\mathbf{NP} \not\subseteq \mathbf{P}_{/\mathbf{poly}}$ and may give insight to prove $\mathbf{NP} \not\subseteq \mathbf{P}_{/\mathbf{poly}}$.

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- **NC** and **AC** correspond to efficient parallel computation.

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PARITY is in $\mathbf{NC}^1$ but not in $\mathbf{AC}^0$.

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